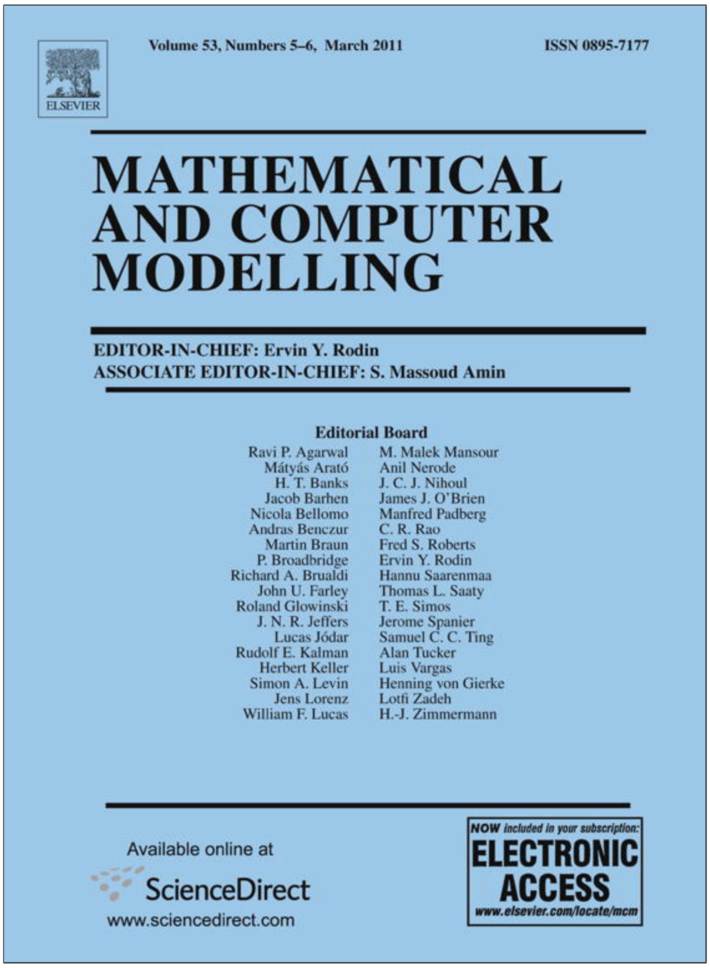
ELSEVIER
Volume 53, Numbers 5–6, March 2011
ISSN 0895-7177
MATHEMATICAL AND COMPUTER MODELLING
EDITOR-IN-CHIEF: Ervin Y. Rodin
ASSOCIATE EDITOR-IN-CHIEF: S. Massoud Amin
Editorial Board
Ravi P. Agarwal
Mátyás Arató
H. T. Banks
Jacob Barhen
Nicola Bellomo
Andras Benczur
Martin Braun
P. Broadbridge
Richard A. Brualdi
John U. Farley
Roland Glowinski
J. N. R. Jeffers
Lucas Jódar
Rudolf E. Kalman
Herbert Keller
Simon A. Levin
Jens Lorenz
William F. Lucas
M. Malek Mansour
Anil Nerode
J. C. J. Nihoul
James J. O'Brien
Manfred Padberg
C. R. Rao
Fred S. Roberts
Ervin Y. Rodin
Hannu Saarenmaa
Thomas L. Saaty
T. E. Simos
Jerome Spanier
Samuel C. C. Ting
Alan Tucker
Luis Vargas
Henning von Gierke
Lotfi Zadeh
H.-J. Zimmermann
Available online at
ScienceDirect
www.sciencedirect.com
NOW included in your subscription:
ELECTRONIC ACCESS
www.elsevier.com/locate/mcm

Mathematical and Computer Modelling 53 (2011) 581–586

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Integral operators on Ma–Minda type starlike and convex functions

Rosihan M. Ali [a,*], V. Ravichandran [b]

[a] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
[b] Department of Mathematics, University of Delhi, Delhi-110007, India

ARTICLE INFO

Article history:
Received 28 January 2010
Received in revised form 14 September 2010
Accepted 14 September 2010

Keywords:
Analytic functions
Differential subordination
Ma–Minda type starlike and convex functions
Integral operators

ABSTRACT

Two integral operators on the classes consisting of normalized p-valent Ma–Minda type starlike and convex functions are considered. Functions in these classes have the form $zf'(z)/f(z) \prec p\varphi(z)$ and $1 + zf''(z)/f'(z) \prec p\varphi(z)$ respectively, where φ is a convex function with $\varphi(0) = 1$. It is shown that the first of these operators maps starlike functions into convex functions, while the convex mappings are shown to be closed under the second integral operator.

1. Introduction and motivation

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane and let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{D}$ and normalized by the conditions $f(0) = 0$, and $f'(0) = 1$. An analytic function f is *subordinate* to an analytic function g, written $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. When the function g is univalent in $\mathbb{D}$, the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is a starlike domain with respect to 0, and a function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is a convex domain. Analytically, these requirements are respectively equivalent to the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

In terms of subordination, these conditions are expressed respectively in the forms

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}.$$

Ma and Minda [1] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $(1 + z)/(1 - z)$ with a more general function φ. This analytic function φ has positive real part with $\varphi(0) = 1$, and maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1. Ma and Minda introduced the following classes that includes several well-known starlike and convex mappings as special cases:

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

* Corresponding author.
E-mail addresses: rosihan@cs.usm.my (R.M. Ali), vravi@maths.du.ac.in (V. Ravichandran).

0895-7177/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2010.09.007

and

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

Let $\mathcal{A}_p$ be the class of all p-valent analytic functions $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$ in the open unit disk $\mathbb{D}$. The class $\mathcal{A}_1$ will be denoted by $\mathcal{A}$. Following Ma and Minda [1], the following classes of p-valent starlike and convex functions were introduced and investigated in [2].

Definition 1 ([2]). Let φ be an analytic univalent function in $\mathbb{D}$ with $\varphi(0) = 1$. The class $\mathcal{CV}_p(\varphi)$ consists of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \quad (z \in \mathbb{D}),$$

and the class $\mathcal{ST}_p(\varphi)$ consists of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in \mathbb{D}).$$

Let $\varphi_\beta : \mathbb{D} \to \mathbb{C}$ be the function defined by

$$\varphi_\beta(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad \beta \neq 1.$$

When $\beta < 1$, $\varphi_\beta(\mathbb{D})$ is the half-plane defined by Re $w > \beta$, while in the case $\beta > 1$, $\varphi_\beta(\mathbb{D})$ is the half-plane defined by Re $w < \beta$. Thus for $\beta < 1$, the classes $\mathcal{ST}_p(\varphi_\beta)$ and $\mathcal{CV}_p(\varphi_\beta)$ reduce to the familiar classes of p-valent starlike and convex functions of order β:

$$\mathcal{ST}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \right\},$$

$$\mathcal{CV}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \right\}.$$

Similarly, for $\beta > 1$, the classes $\mathcal{ST}_p(\varphi_\beta)$ and $\mathcal{CV}_p(\varphi_\beta)$ reduce respectively to the equivalent classes

$$\mathcal{M}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \right\},$$

$$\mathcal{N}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta \right\}.$$

For $p = 1$, these classes were considered by Breaz [3], Nishiwaki and Owa [4], Owa and Nishiwaki [5], Owa and Srivastava [6], and Uralegaddi et al. [7].

Next let $\varphi_{\lambda,\mu} : \mathbb{D} \to \mathbb{C}$ be the conformal mapping of $\mathbb{D}$ onto the domain

$$\Omega_{\lambda,\mu} = \{w \in \mathbb{C} : \mathrm{Re}\, w - \mu \geq \lambda|w - 1|\},$$

and normalized by $\varphi_{\lambda,\mu}(0) = 1$. Then the classes $\mathcal{ST}_p(\varphi_{\lambda,\mu})$ and $\mathcal{CV}_p(\varphi_{\lambda,\mu})$ reduce to the classes $\mathcal{ST}_p(\lambda, \mu)$ and $\mathcal{CV}_p(\lambda, \mu)$ of p-valent starlike and convex functions associated with parabolic starlike and uniformly convex functions. The class $\mathcal{CV}_p(\lambda, \mu)$ was investigated by Yang and Owa [8], and Frasin [9]. In fact the classes $\mathcal{C}_p(\lambda, \mu)$ and $\mathcal{UC}_p(\beta, k)$ investigated by Frasin [9] are essentially the same: $\mathcal{C}_p(\lambda, \mu) = \mathcal{UC}_p(p\mu, \lambda)$. We shall consider only the former class in this paper, which in our notation is the class $\mathcal{CV}_p(\lambda, \mu)$.

For $\alpha_i \geq 0$ and $f_i \in \mathcal{A}_p$, define the following respective integral operators:

$$F_p(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t^p}\right)^{\alpha_i} \mathrm{d}t, \tag{1.1}$$

$$G_p(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left(\frac{f_i'(t)}{pt^{p-1}}\right)^{\alpha_i} \mathrm{d}t. \tag{1.2}$$

In this paper, the above defined integral operators are investigated for the classes of p-valent Ma–Minda type starlike and convex functions. It is shown that F_p defined by (1.1) transforms a Ma–Minda type starlike function into a Ma–Minda type convex function. It is also shown that the Ma–Minda type convex functions are closed under the operator G_p given by (1.2). In the special case $p = 1$, the results obtained here include several earlier works found in the literature.

2. Convexity of the integral operators

Theorem 2.1. *Let $\alpha_i \geq 0$, and $f_i \in \mathcal{A}_p$, $i = 1, 2, \ldots, n$. Let F_p be given by (1.1).*

(1) *If $f_i \in \mathcal{ST}_p(\beta_i)$, $\beta_i < 1$, then $F_p \in \mathcal{CV}_p(\gamma)$ where $\gamma := 1 - \sum_{i=1}^n \alpha_i(1-\beta_i)$. In particular, if $\sum_{i=1}^n \alpha_i(1-\beta_i) \leq 1$, then $F_p \in \mathcal{CV}_p := \mathcal{CV}_p(0)$.*
(2) *If $f_i \in \mathcal{M}_p(\beta_i)$, $\beta_i > 1$, then $F_p \in \mathcal{N}_p(\gamma)$ where $\gamma := 1 + \sum_{i=1}^n \alpha_i(\beta_i - 1)$.*

Proof. Since

$$F_p'(z) = pz^{p-1}\prod_{i=1}^n \left(\frac{f_i(z)}{z^p}\right)^{\alpha_i},$$

it follows that

$$\frac{1}{p}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) = \left(1 - \sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i \frac{1}{p}\left(\frac{zf_i'(z)}{f_i(z)}\right).$$

The desired results are now evident from the definitions of the above classes. □

Corollary 2.1. *Let $\alpha_i \geq 0$, and $f_i \in \mathcal{A}_p$, $i = 1, 2, \ldots, n$. Let F_p be given by (1.1).*

(1) *If $f_i \in \mathcal{ST}_p(\beta)$, $\beta < 1$, then $F_p \in \mathcal{CV}_p(\gamma)$ where $\gamma := 1-(1-\beta)\sum_{i=1}^n \alpha_i$. In particular, if $\sum_{i=1}^n \alpha_i \leq 1$, then $F_p \in \mathcal{CV}_p(\beta)$.*
(2) *If $f_i \in \mathcal{M}_p(\beta)$, $\beta > 1$, then $F_p \in \mathcal{N}_p(\gamma)$ where $\gamma := 1 + (\beta - 1)\sum_{i=1}^n \alpha_i$.*

Given a complex number $b \neq 0$, the classes of p-valent starlike and convex functions of complex order b and type β ($\beta < 1$), are defined as below:

$$\mathcal{ST}_p(b, \beta) := \left\{f \in \mathcal{A}_p : \operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right) > \beta\right\},$$

$$\mathcal{CV}_p(b, \beta) := \left\{f \in \mathcal{A}_p : \operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right)\right) > \beta\right\}.$$

For $p = 1$, these classes were considered by [10–13]. It is clear that $\mathcal{ST}_p(b, \beta) = \mathcal{ST}_p(b(1-\beta), 0)$ and $\mathcal{CV}_p(b, \beta) = \mathcal{CV}_p(b(1-\beta), 0)$. Similarly, for $\beta > 1$, we define the following classes:

$$\mathcal{M}_p(b, \beta) := \left\{f \in \mathcal{A}_p : \operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right) < \beta\right\},$$

$$\mathcal{N}_p(b, \beta) := \left\{f \in \mathcal{A}_p : \operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right)\right) < \beta\right\}.$$

Theorem 2.1 extends to the above defined classes as shown in the following result:

Theorem 2.2. *Let $\alpha_i \geq 0$, and $f_i \in \mathcal{A}_p$, $i = 1, 2, \ldots, n$. Let F_p be given by (1.1).*

(1) *If $f_i \in \mathcal{ST}_p(b, \beta_i)$, $\beta_i < 1$, then $F_p \in \mathcal{CV}_p(b, \gamma)$ where $\gamma := 1 - \sum_{i=1}^n \alpha_i(1-\beta_i)$.*
(2) *If $f_i \in \mathcal{M}_p(b, \beta_i)$, $\beta_i > 1$, then $F_p \in \mathcal{N}_p(b, \gamma)$ where $\gamma := 1 + \sum_{i=1}^n \alpha_i(\beta_i - 1)$.*

Proof. The result follows by noting that

$$1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) - 1\right) = \left(1 - \sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf_i'(z)}{f_i(z)} - 1\right)\right). \quad \square$$

Remark 2.1. Theorem 2.2(1) extends the work of Bulut [12]. In particular, when $p = 1$, Theorem 2.2(1) reduces to Theorem 1 in [12].

Theorem 2.3. *Let $\alpha_i \geq 0$, and $f_i \in \mathcal{A}_p$, $i = 1, 2, \ldots, n$. Let G_p be given by (1.2).*

(1) *If $f_i \in \mathcal{CV}_p(\beta_i)$, $\beta_i < 1$, then $G_p \in \mathcal{CV}_p(\gamma)$ where $\gamma := 1 - \sum_{i=1}^n \alpha_i(1-\beta_i)$. In particular, if $\sum_{i=1}^n \alpha_i(1-\beta_i) \leq 1$, then $G_p \in \mathcal{CV}_p$.*
(2) *If $f_i \in \mathcal{N}_p(\beta_i)$, $\beta_i > 1$, then $G_p \in \mathcal{N}_p(\gamma)$ where $\gamma := 1 + \sum_{i=1}^n \alpha_i(\beta_i - 1)$.*

Proof. Since

$$G_p'(z) = pz^{p-1} \prod_{i=1}^{n} \left(\frac{f_i'(z)}{pz^{p-1}} \right)^{\alpha_i},$$

it follows that

$$\frac{1}{p}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) = \left(1 - \sum_{i=1}^{n} \alpha_i\right) + \sum_{i=1}^{n} \alpha_i \frac{1}{p}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right).$$

The desired results follow directly from the definitions of the classes. □

Corollary 2.2. *Let $\alpha_i \geq 0$, and $f_i \in \mathcal{A}_p$, $i = 1, 2, \ldots, n$. Let G_p be given by (1.2).*

(1) *If $f_i \in \mathcal{CV}_p(\beta)$, $\beta < 1$, then $G_p \in \mathcal{CV}_p(\gamma)$ where $\gamma := 1 - (1 - \beta)\sum_{i=1}^{n} \alpha_i$. In particular, if $\sum_{i=1}^{n} \alpha_i \leq 1$, then $G_p \in \mathcal{CV}_p(\beta)$.*
(2) *If $f_i \in \mathcal{N}_p(\beta)$, $\beta > 1$, then $G_p \in \mathcal{N}_p(\gamma)$ where $\gamma := 1 + (\beta - 1)\sum_{i=1}^{n} \alpha_i$.*

In general, the following result is obtained:

Theorem 2.4. *Let $\alpha_i \geq 0$, and $f_i \in \mathcal{A}_p$, $i = 1, 2, \ldots, n$. Let G_p be given by (1.2).*

(1) *If $f_i \in \mathcal{CV}_p(b, \beta_i)$, $\beta_i < 1$, then $G_p \in \mathcal{CV}_p(b, \gamma)$ where $\gamma := 1 - \sum_{i=1}^{n} \alpha_i(1 - \beta_i)$.*
(2) *If $f_i \in \mathcal{N}_p(b, \beta_i)$, $\beta_i > 1$, then $G_p \in \mathcal{N}_p(b, \gamma)$ where $\gamma := 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$.*

Proof. The results follow from the equation

$$1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) - 1\right) = \left(1 - \sum_{i=1}^{n} \alpha_i\right) + \sum_{i=1}^{n} \alpha_i \left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) - 1\right)\right). \quad \square$$

Remark 2.2. For $p = 1$, Theorem 2.2(1) reduces to Theorem 3 in [12].

As applications of our results, the following results are obtained for the class $\mathcal{CV}_p(\lambda, \mu)$.

Theorem 2.5. *For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$, $\mu_i \geq \lambda_i \geq 0$ and*

$$\gamma := 1 - \sum_{i=1}^{n} \alpha_i \frac{1 - \mu_i}{1 + \lambda_i}.$$

(1) *If $f_i \in \mathcal{SP}_p(\lambda_i, \mu_i)$, then $F_p \in \mathcal{CV}_p(\gamma)$.*
(2) *If $f_i \in \mathcal{CV}_p(\lambda_i, \mu_i)$, then $G_p \in \mathcal{CV}_p(\gamma)$.*

Proof. We first prove that $\mathcal{ST}_p(\lambda, \mu) \subset \mathcal{ST}_p((\mu + \lambda)/(1 + \lambda))$. Let $f \in \mathcal{ST}_p(\lambda, \mu)$. Then the quantity $w := zf'(z)/(pf(z))$ satisfies

$$\operatorname{Re} w - \mu \geq \lambda|w - 1|.$$

The inequality

$$\operatorname{Re} w - \mu \geq -\lambda \operatorname{Re}(w - 1)$$

yields

$$\operatorname{Re} w \geq \frac{\mu + \lambda}{1 + \lambda}.$$

Thus $f \in \mathcal{ST}_p\left(\frac{\mu+\lambda}{1+\lambda}\right)$. Now since $f_i \in \mathcal{ST}_p(\lambda_i, \mu_i)$, then $f_i \in \mathcal{ST}_p((\mu_i + \lambda_i)/(1 + \lambda_i))$, and the results of the theorem now follows from an application of Theorem 2.1(1).

The proof of the second part of the theorem follows similarly from Theorem 2.3(1). □

Remark 2.3. Since

$$1 - \sum_{i=1}^{n} \frac{\alpha_i(1 - \mu_i)}{1 + \lambda_i} \geq 1 - \sum_{i=1}^{n} \alpha_i(1 - \mu_i),$$

Theorem 2.5(2) improves the corresponding result of Frasin [9, Theorem 3.6]. It should be pointed out that the result obtained by Frasin is independent of the parameters λ_i, where as these parameters play an important role in our Theorem 2.5(2).

Next let $-1 \leq B \leq A \leq 1$, and $\varphi_{A,B}$ be given by

$$\varphi_{A,B}(z) = \frac{1+Az}{1+Bz} \quad (z \in \mathbb{D}).$$

Let $\mathcal{ST}_p(A,B) := \mathcal{ST}_p(\varphi_{A,B})$ and $\mathcal{CV}_p(A,B) := \mathcal{CV}_p(\varphi_{A,B})$. It can be shown that

$$\mathcal{ST}_p(A,B) \subset \mathcal{ST}_p((1-A)/(1-B)).$$

Using this fact, the following theorem is evident:

Theorem 2.6. *Let $\alpha_i \geq 0$, $-1 < B_i < A_i \leq 1$, $i = 1, 2, \ldots, n$, and*

$$\gamma := 1 - \sum_{i=1}^{n} \alpha_i \frac{A_i - B_i}{1 - B_i}.$$

(1) *If $f_i \in \mathcal{ST}_p(A_i, B_i)$, then $F_p \in \mathcal{CV}_p(\gamma)$.*
(2) *If $f_i \in \mathcal{CV}_p(A_i, B_i)$, then $G_p \in \mathcal{CV}_p(\gamma)$.*

3. Closure property of integral operators

For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$, $\beta < 1$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. For $f_i \in \mathcal{A}_p$, let F_p be given by (1.1). By Corollary 2.1, if $f_i \in \mathcal{ST}_p(\beta)$, then $F_p \in \mathcal{CV}_p(\beta)$. We prove this in a more general setting in the following theorem:

Theorem 3.1. *For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Let φ be convex in $\mathbb{D}$ with $\varphi(0) = 1$. If $f_i \in \mathcal{ST}_p(\varphi)$, then $F_p \in \mathcal{CV}_p(\varphi)$.*

Proof. As shown in the proof of Theorem 2.1, it follows that

$$\frac{1}{p}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) = \left(1 - \sum_{i=1}^{n} \alpha_i\right) + \sum_{i=1}^{n} \alpha_i \frac{1}{p}\left(\frac{zf_i'(z)}{f_i(z)}\right).$$

The assumption that $f_i \in \mathcal{ST}_p(\varphi)$, yields

$$\frac{1}{p}\frac{zf_i'(z)}{f_i(z)} \prec \varphi(z),$$

and thus

$$\frac{1}{p}\frac{zf_i'(z)}{f_i(z)} \in \varphi(\mathbb{D}),$$

for every $z \in \mathbb{D}$. Since φ is convex, the convex combination of 1 and $\frac{1}{p}\frac{zf_i'(z)}{f_i(z)}$ $(i = 1, 2, \ldots, n)$, is again in $\varphi(\mathbb{D})$. This shows that

$$\frac{1}{p}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) = \left(1 - \sum_{i=1}^{n} \alpha_i\right)(1) + \sum_{i=1}^{n} \alpha_i \frac{1}{p}\left(\frac{zf_i'(z)}{f_i(z)}\right) \in \varphi(\mathbb{D}),$$

or

$$\frac{1}{p}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) \prec \varphi(z). \quad \square$$

Shanmugam and Ravichandran [14] have shown that if the f_i's are uniformly convex functions and α_i's are real numbers such that $\alpha_i \geq 0$, and $\sum_1^n \alpha_i \leq 1$, then the function

$$\int_0^z \prod_{i=1}^{n} [f_i'(\zeta)]^{\alpha_i} \mathrm{d}\zeta$$

is also uniformly convex. This result was extended to parabolic starlike functions of order ρ by Aghalary and Kulkarni [15]. This result is indeed valid even for a more general class of functions:

Theorem 3.2. *For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Let φ be convex in $\mathbb{D}$ with $\varphi(0) = 1$. If $f_i \in \mathcal{CV}_p(\varphi)$, then $G_p \in \mathcal{CV}_p(\varphi)$.*

The proof is similar to Theorem 3.1, and is therefore omitted.

Remark 3.1. For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Let φ be convex in $\mathbb{D}$ with $\varphi(0) = 1$. If $f_i \in \mathcal{CV}_p(\varphi)$, then it follows from Theorem 3.2 that

$$z^p \prod_{i=1}^{n} \left(\frac{f_i'(z)}{pz^{p-1}} \right)^{\alpha_i} \in \mathcal{ST}_p(\varphi).$$

Acknowledgements

The work presented here is supported by research grants from Universiti Sains Malaysia and University of Delhi.

References

[1] W.C. Ma, D. Minda, A unified treatment of some special classes of univalent functions, in: Proceedings of the Conference on Complex Analysis, Tianjin, Internat. Press, Cambridge, MA, 1994, pp. 157–169.
[2] R.M. Ali, V. Ravichandran, S.K. Lee, Subclasses of multivalent starlike and convex functions, Bull. Belg. Math. Soc. Simon Stevin 16 (3) (2009) 385–394.
[3] D. Breaz, Certain integral operators on the classes $\mathbb{M}(\beta_i)$ and $\mathbb{N}(\beta_i)$, J. Inequal. Appl. (2008) 4pp. Art. ID 719354.
[4] J. Nishiwaki, S. Owa, Coefficient inequalities for analytic functions, Int. J. Math. Math. Sci. 29 (5) (2002) 285–290.
[5] S. Owa, J. Nishiwaki, Coefficient estimates for certain classes of analytic functions, JIPAM. J. Inequal. Pure Appl. Math. 3 (5) (2002) 5 pp, Article 72 (electronic).
[6] S. Owa, H.M. Srivastava, Some generalized convolution properties associated with certain subclasses of analytic functions, JIPAM. J. Inequal. Pure Appl. Math. 3 (3) (2002) 13 pp, Article 42 (electronic).
[7] B.A. Uralegaddi, M.D. Ganigi, S.M. Sarangi, Univalent functions with positive coefficients, Tamkang J. Math. 25 (3) (1994) 225–230.
[8] D. Yang, S. Owa, Properties of certain p-valently convex functions, Int. J. Math. Math. Sci. 41 (2003) 2603–2608.
[9] B.A. Frasin, Convexity of integral operators of p-valent functions, Math. Comput. Modelling 51 (5–6) (2010) 601–605.
[10] D. Breaz, H.Ö. Guney, The integral operator on the classes $\mathcal{S}_\alpha^*(b)$ and $\mathcal{C}_\alpha(b)$, J. Math. Inequal. 2 (1) (2008) 97–100.
[11] D. Breaz, M.K. Aouf, N. Breaz, Some properties for integral operators on some analytic functions with complex order, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 25 (2009) 39–43.
[12] S. Bulut, A note on the paper of Breaz and Gňey, J. Math. Inequal. 2 (4) (2008) 549–553.
[13] B.A. Frasin, Family of analytic functions of complex order, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 22 (2006) 179–191.
[14] T.N. Shanmugam, V. Ravichandran, Certain properties of uniformly convex functions, in: Computational Methods and Function Theory (Penang), World Sci. Publ., River Edge, NJ, 1994, pp. 319–324.
[15] R. Aghalary, S.R. Kulkarni, Certain properties of parabolic starlike and convex functions of order ρ, Bull. Malays. Math. Sci. Soc. (2) 26 (2) (2003) 153–162.